Convolution

In the spatial domain

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \tilde{f}(x)$$

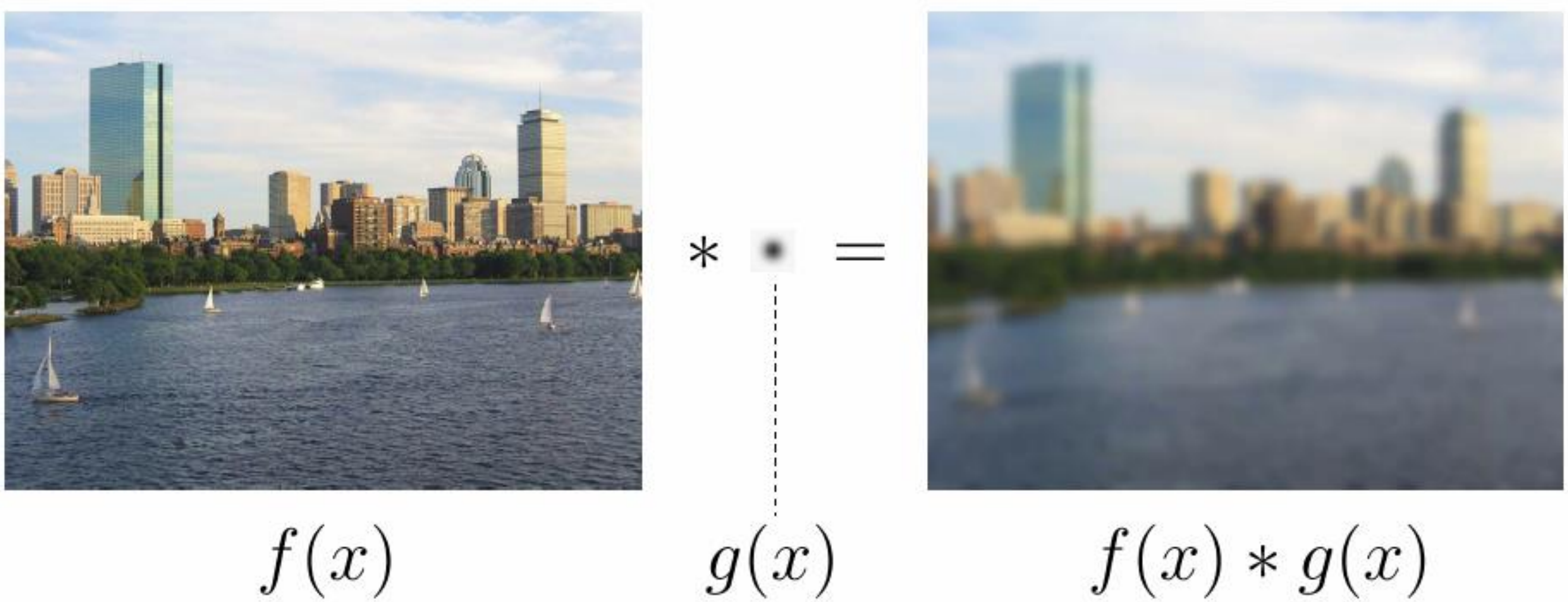
Convolution kernel, filter $g(x)$

Filtered signal $\tilde{f}(x)$

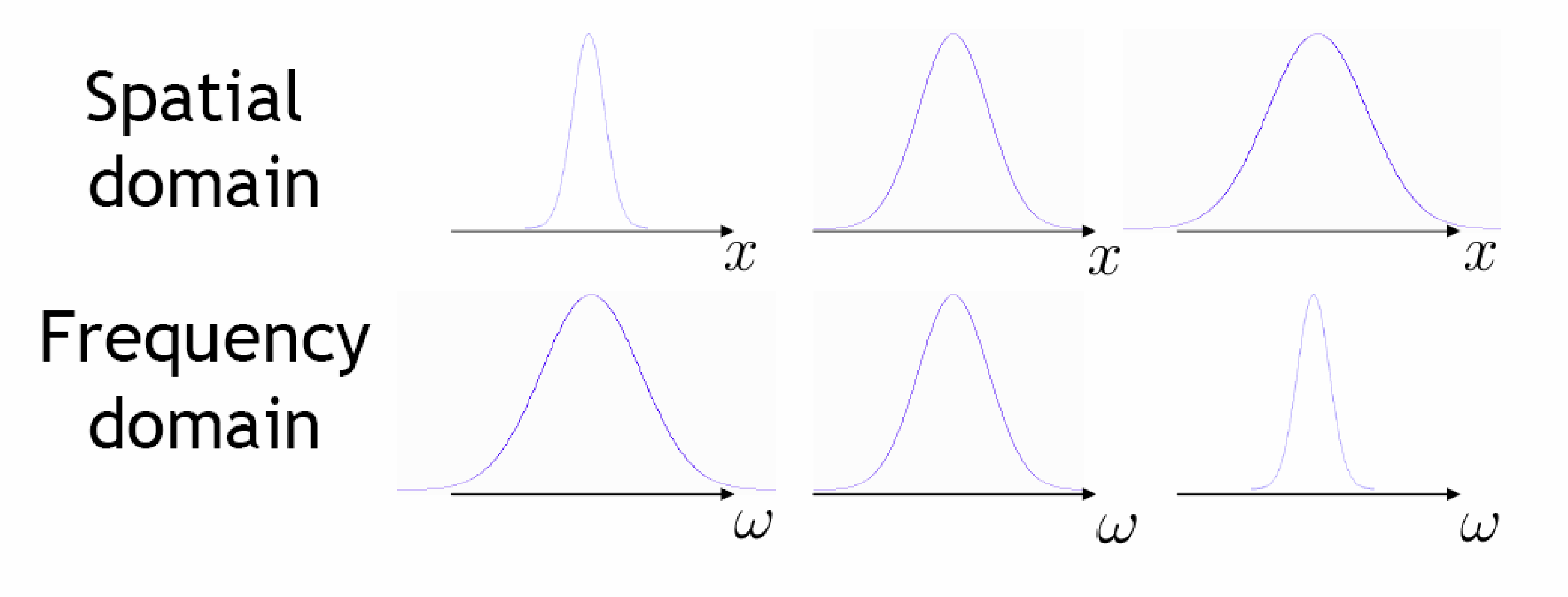
Convolution

- In the spatial domain

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \tilde{f}(x)$$



Convolution



Теорема А.2 (ФУБИНИ). *Если* $\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|f(x_1,x_2)|dx_1\right)dx_2 < +\infty$, *то*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1,x_2)dx_1dx_2 = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x_1,x_2)dx_1\right)dx_2 = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x_1,x_2)dx_2\right)dx_1.$$

Теорема о преобразовании Фурье свертки

Теорема. (О свёртке) Пусть $f \in L^1(R)$ и $h \in L^1(R)$. Функция $g = h * f$ принадлежит $L^1(R)$ и

$$\hat{g}(\omega) = \hat{h}(\omega)\hat{f}(\omega) \quad (\hat{g}\text{— преобр. Фурье g})$$

Доказательство:

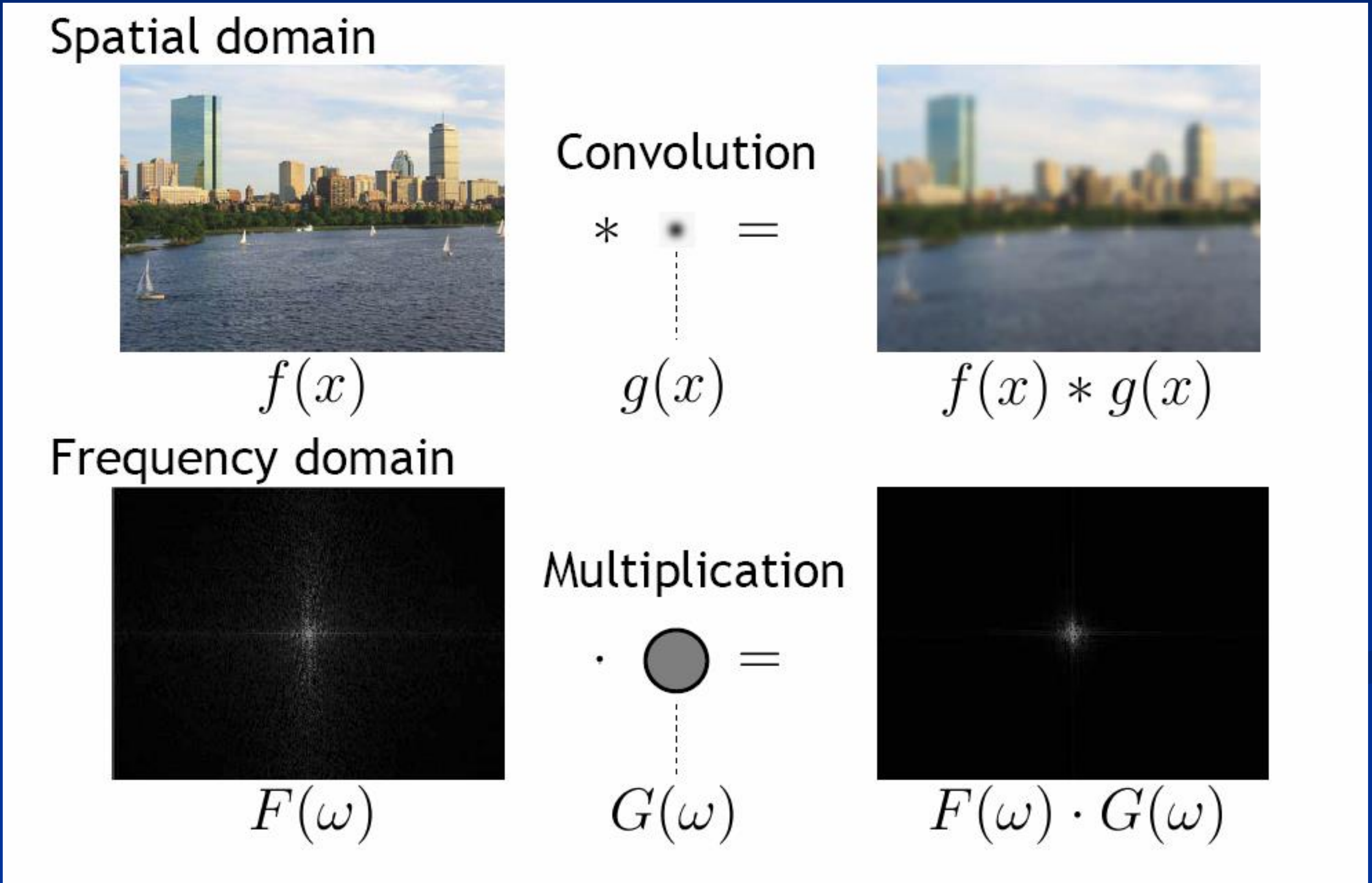
$$\hat{g}(\omega) = \int_{-\infty}^{+\infty} \exp(-it\omega) \left(\int_{-\infty}^{+\infty} f(t-u)h(u)\mathrm{d}u \right) \mathrm{d}t$$

Так как $|f(t-u)||h(u)|$ интегрируема в R^2, мы можем применить теорему Фубини, и замена переменных $(t, u) \to (v = t - u, u)$ даёт

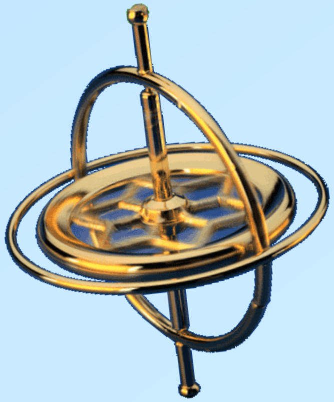
$$\hat{g}(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(-i(u+v)\omega) f(v)h(u)\mathrm{d}u\mathrm{d}v = \left(\int_{-\infty}^{+\infty} \exp(-iv\omega) f(v)\mathrm{d}v \right) \left(\int_{-\infty}^{+\infty} \exp(-iu\omega) h(u)\mathrm{d}u \right)$$

теорема доказана.

Convolution



Removing motion blur from a single image



Sources of blur

- Object motion

Sources of blur

- Object motion

- Translation of camera

Sources of blur

- Object motion

- Translation of camera

- Rotation of camera

Sources of blur

- Object motion

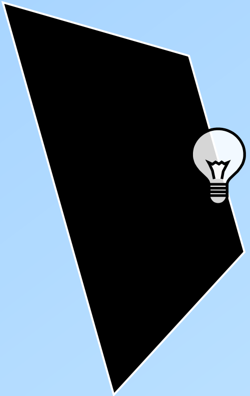
- Translation of camera

- Rotation of camera

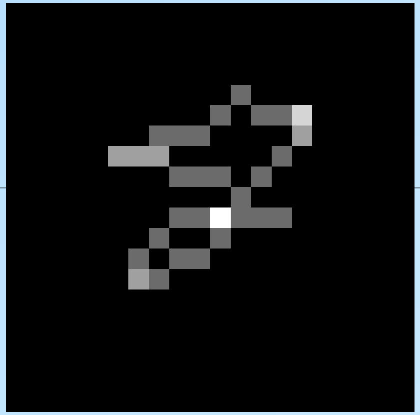
Point Spread Function (PSF)

Assume:

- Point light source



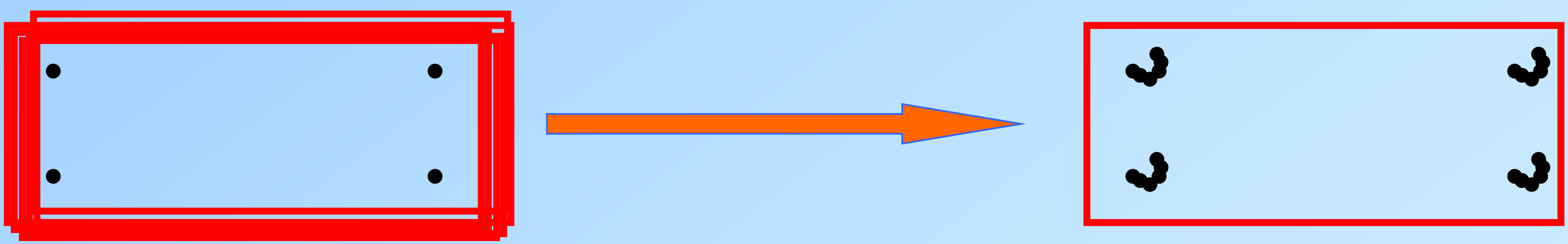
PSF =



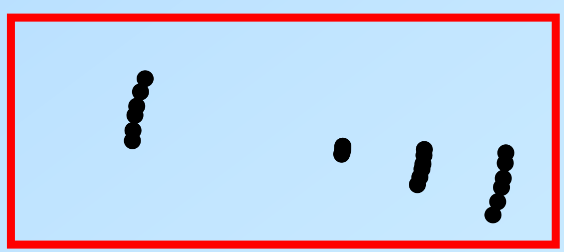
Convolution model motivation

- Assume:
  - No image plane rotation
  - No object motion during the exposure
  - No significant parallax (depth variation)

→ ***Camera motion is Pure Translation!!!***



- Violation of assumption:

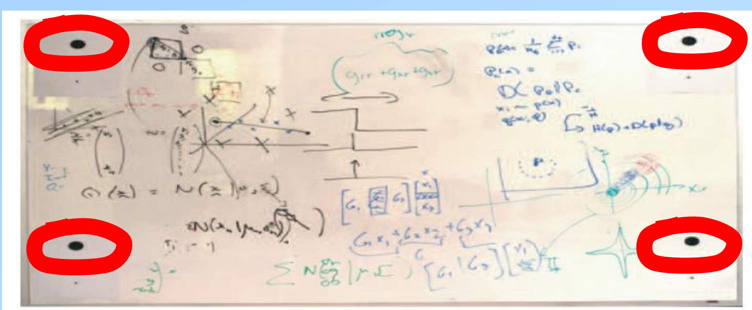


Convolution model motivation

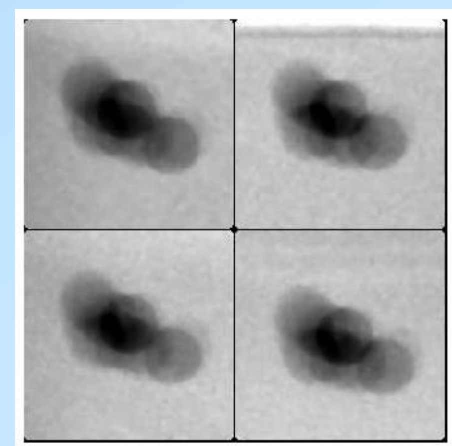
- Assume:
  - No image plane rotation
  - No object motion during the exposure
  - No significant parallax (depth variation)

→ ***Camera motion is Pure Translation!!!***

- Experimental validation:



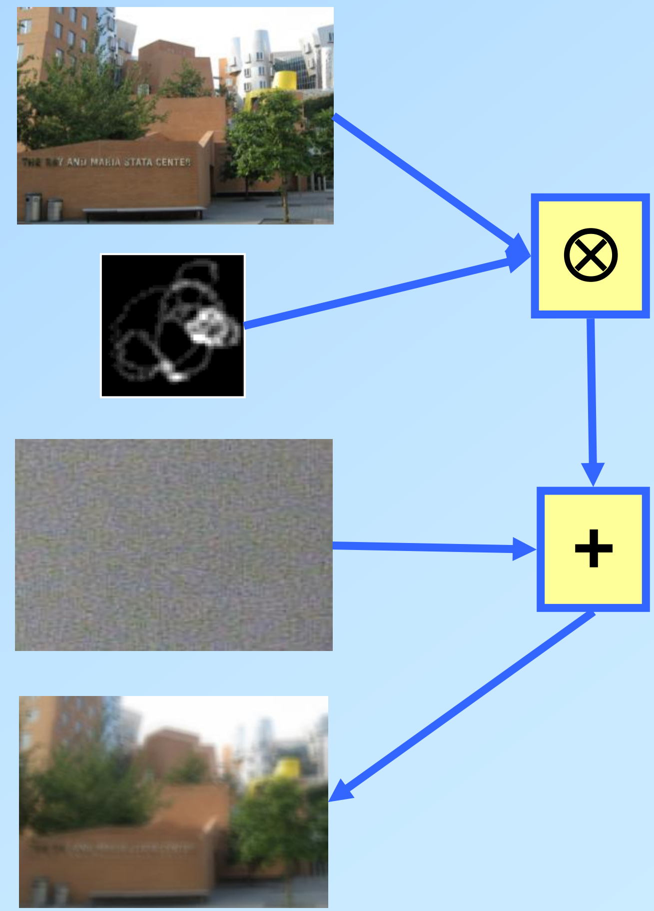
8 subjects handholding DSLR
with 1 sec exposure



Close-up of dots

Convolution Model

- Notations
  - L: original image
  - K: the blur kernel (PSF)
  - N: sensor noise (white)
  - B: input blurred image



Generation rule: $B = K \otimes L + N$

How can the image be recovered?

Goal:

- Recover L s.t.:

$$B = K \otimes L$$

Assumptions:

- Known kernel (PSF)
- Constant kernel for the whole image
- No noise

Fourier Convolution Theorem!

De-blur using Convolution Theorem

Convolution Theorem: $\Im[f \otimes g] = \Im[f] \cdot \Im[g]$

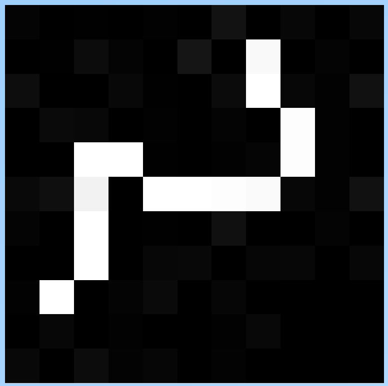
$B = L \otimes K \Rightarrow \quad \Im[B] = \Im[L \otimes K] \Rightarrow$

$\Im[B] = \Im[L] \cdot \Im[K] \Rightarrow \quad \Im[L] = \Im[B] / \Im[K] \Rightarrow$

$$L = \Im^{-1}\left[\Im[B] / \Im[K]\right]$$

Example:

PSF



Blurred Image

Recovered

Noisy case:

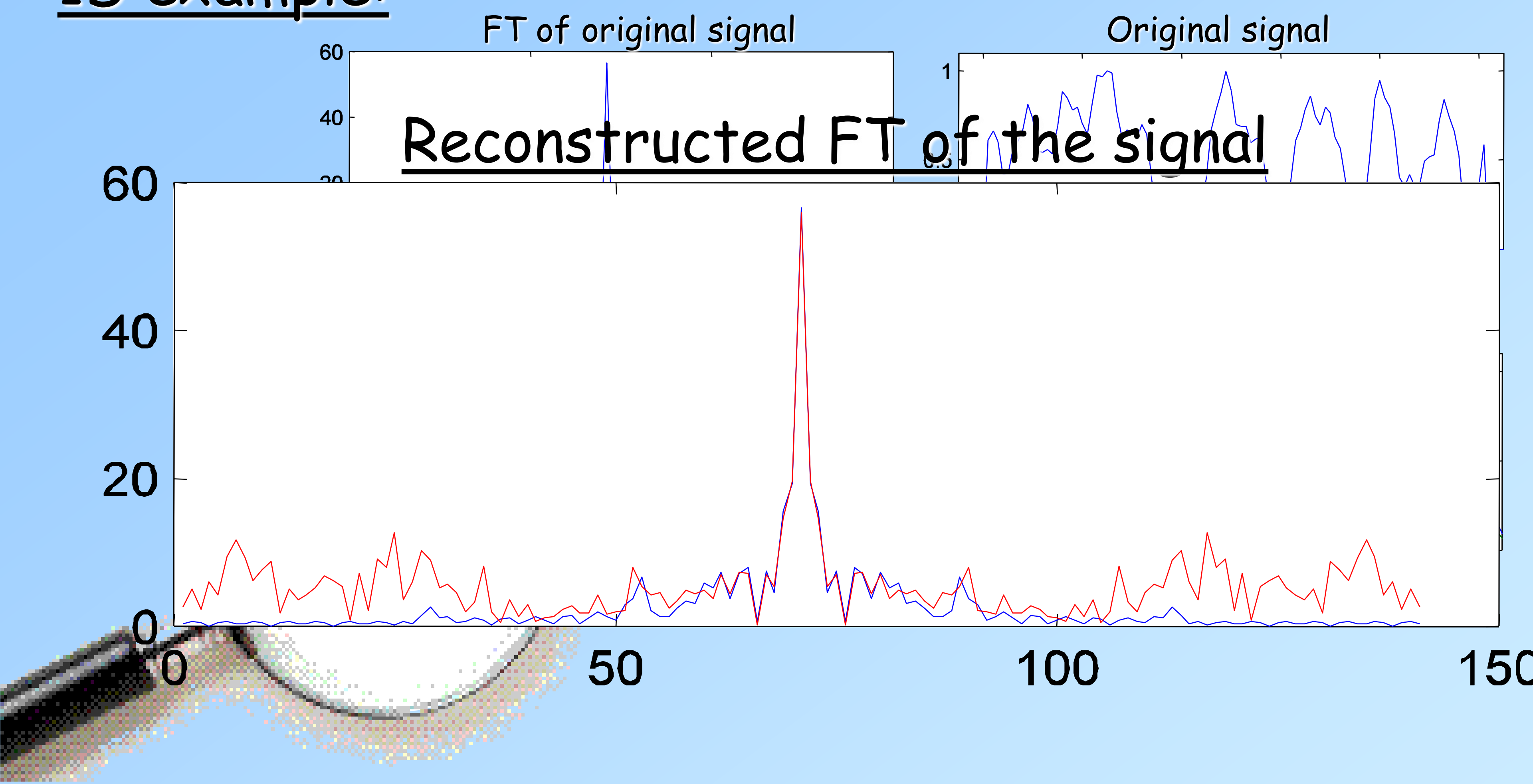
$$L = \Im^{-1}\left[\Im[B]/\Im[K] - \Im[N]/\Im[K]\right]$$

Example:

$\mu = 0, \sigma = 0.001$

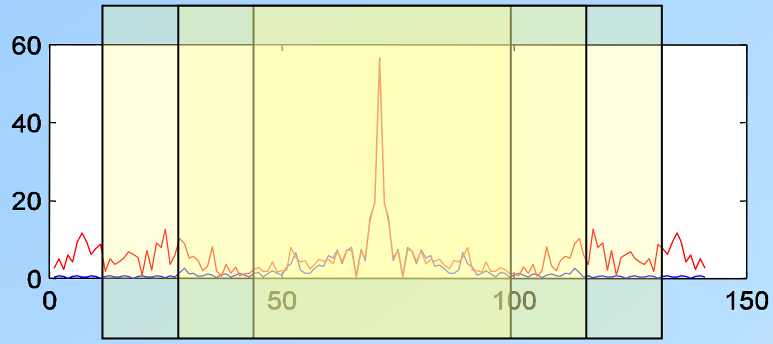
Deconvolution is unstable

1D example:



Regularization is required

Regularizing by window

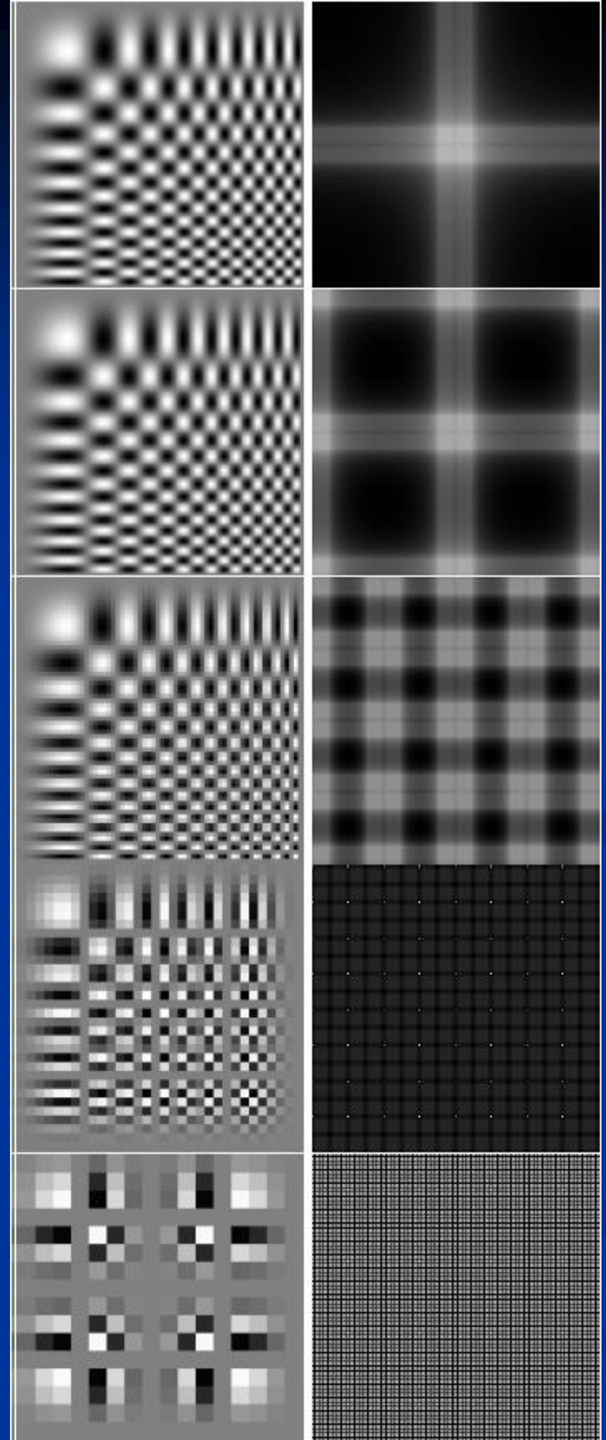


Window size:

51

151

191



Aliasing →

Dirac delta function

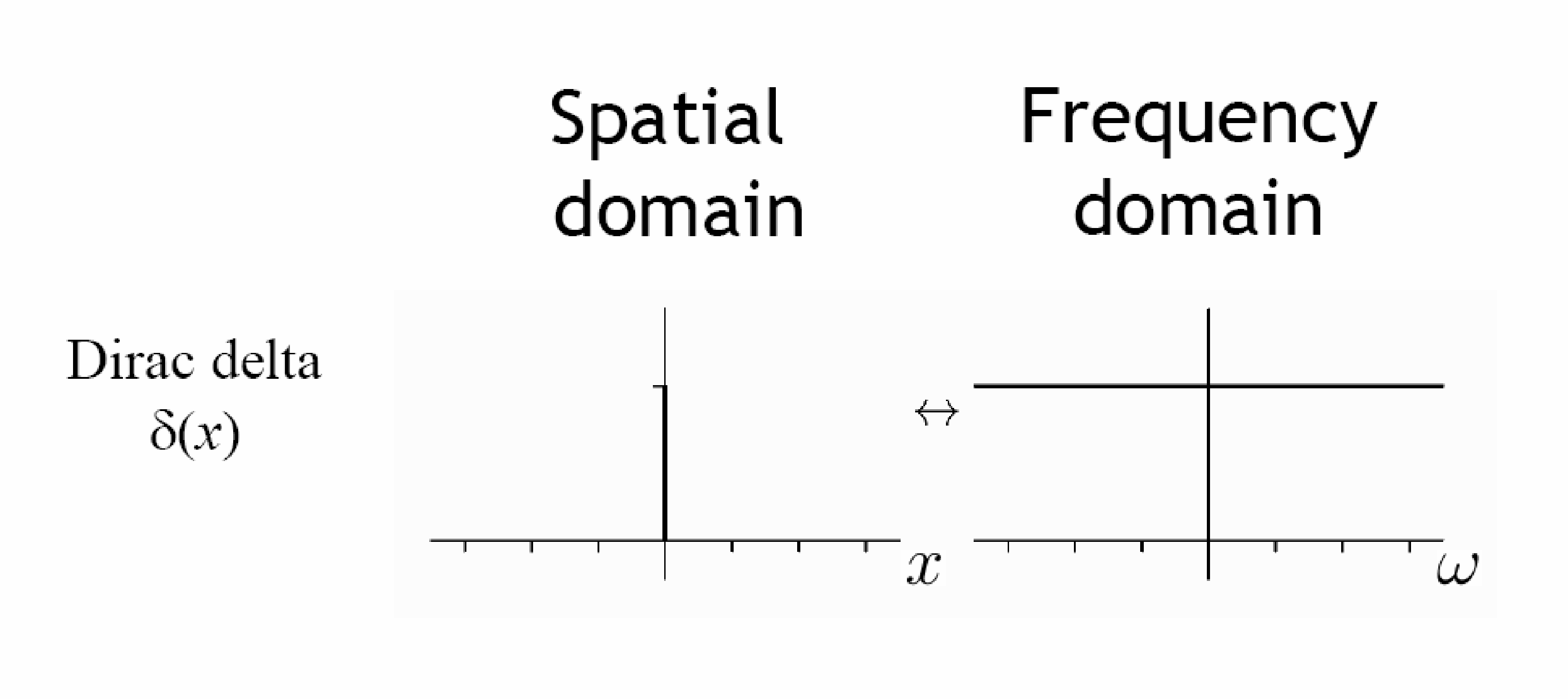
- Definition

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

- Sifting property

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,dx = f(x_0)$$

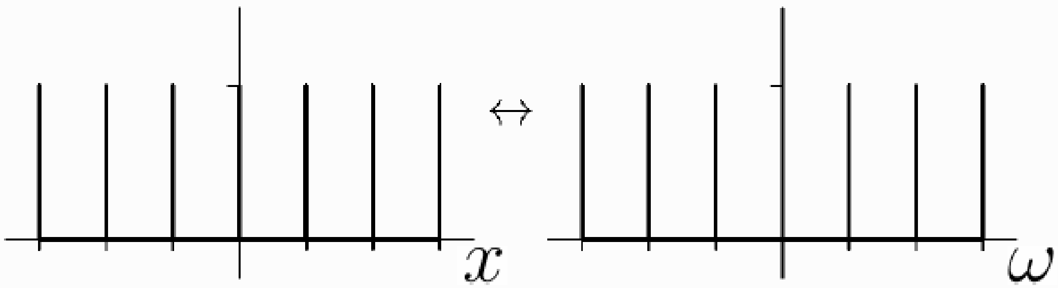
Dirac delta function



Impulse Train

Spatial domain | Frequency domain

Impulse train
(shah, comb function)
$\text{III}_T(x)$



$$\text{III}_T(x) = \sum_k \delta(x - kT) \qquad \text{III}_{\omega_0}(\omega) = \frac{1}{\sqrt{2\pi\omega_0}} \sum_k \delta(\omega - k\omega_0)$$

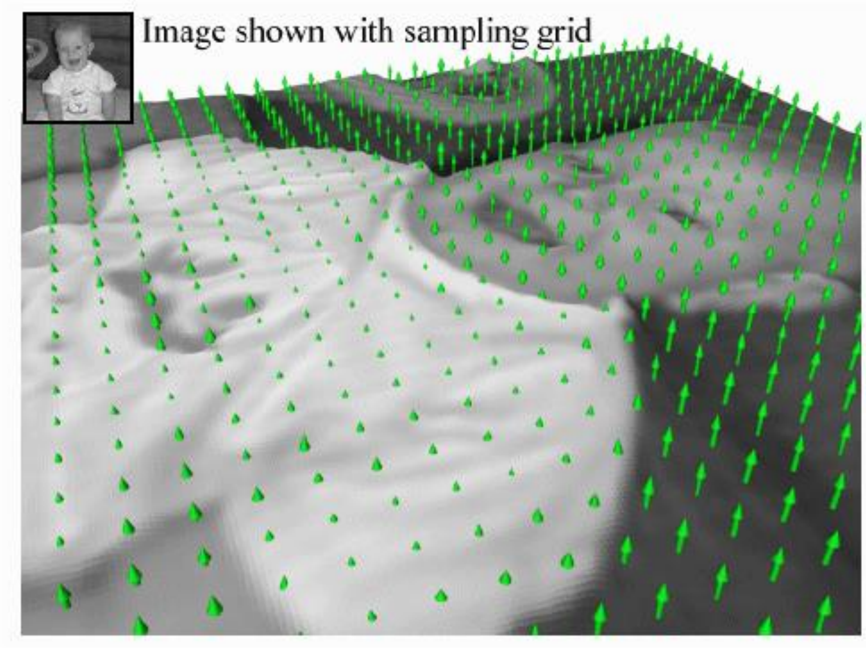
Period T

Period $\omega_0 = 2\pi / T$

Sampling

Spatial domain: multiply signal with impulse train

$$f(x) \rightarrow f(x) III_T(x)$$



Sampling

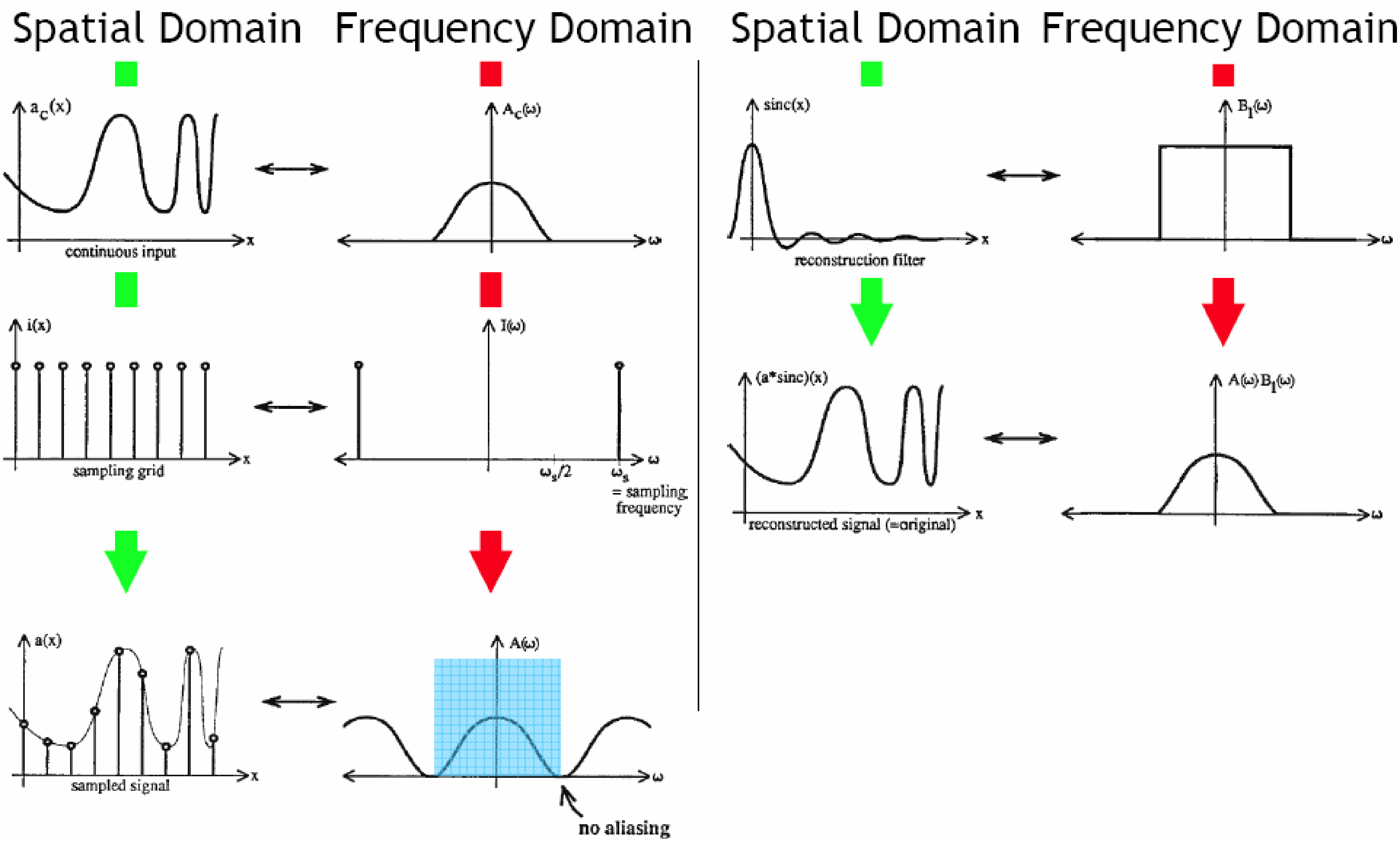
Spatial domain: multiply signal with impulse train

$$f(x) \rightarrow f(x) III_T(x)$$

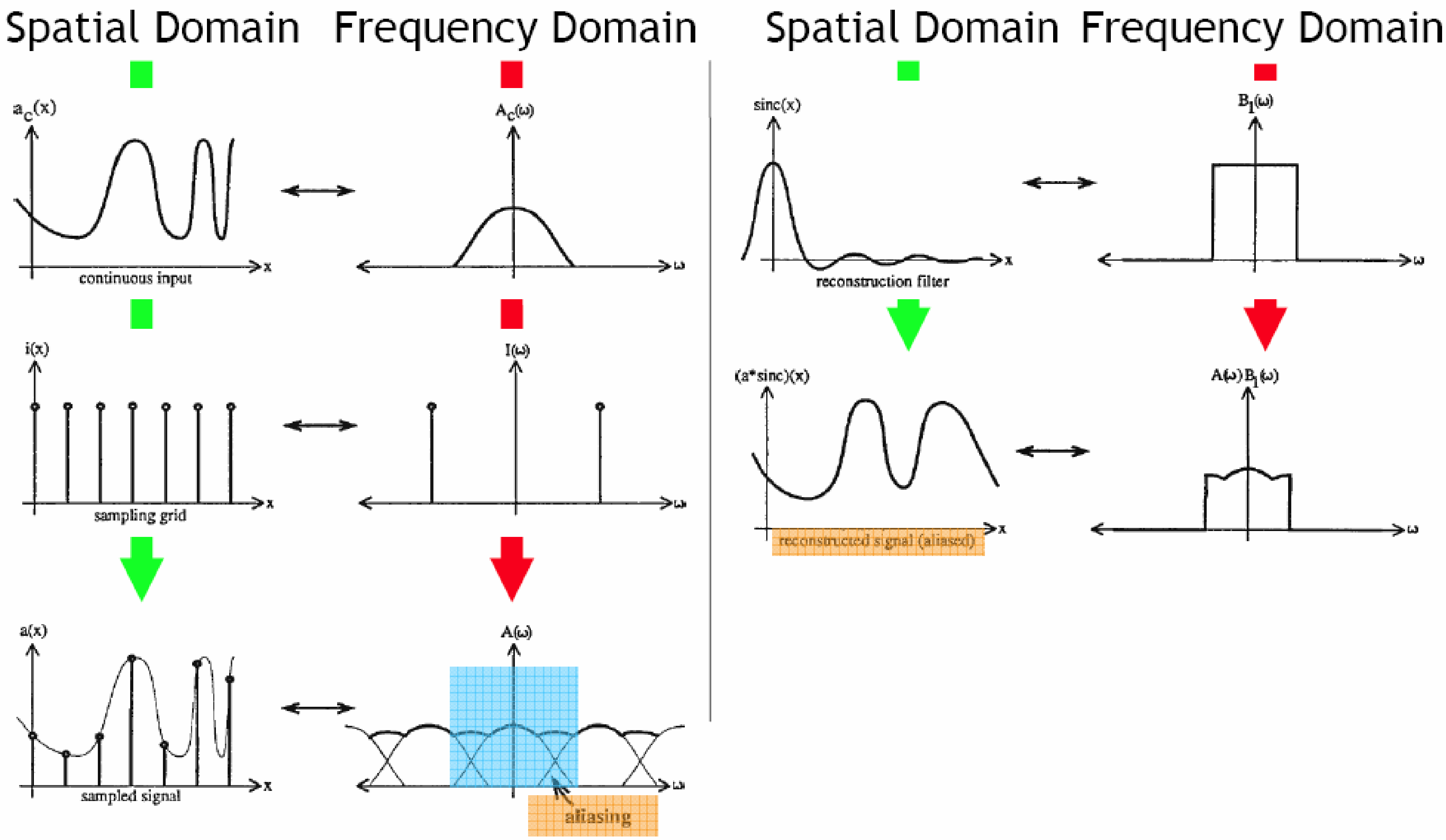
Frequency domain: convolve signal with Fourier transform of impulse train

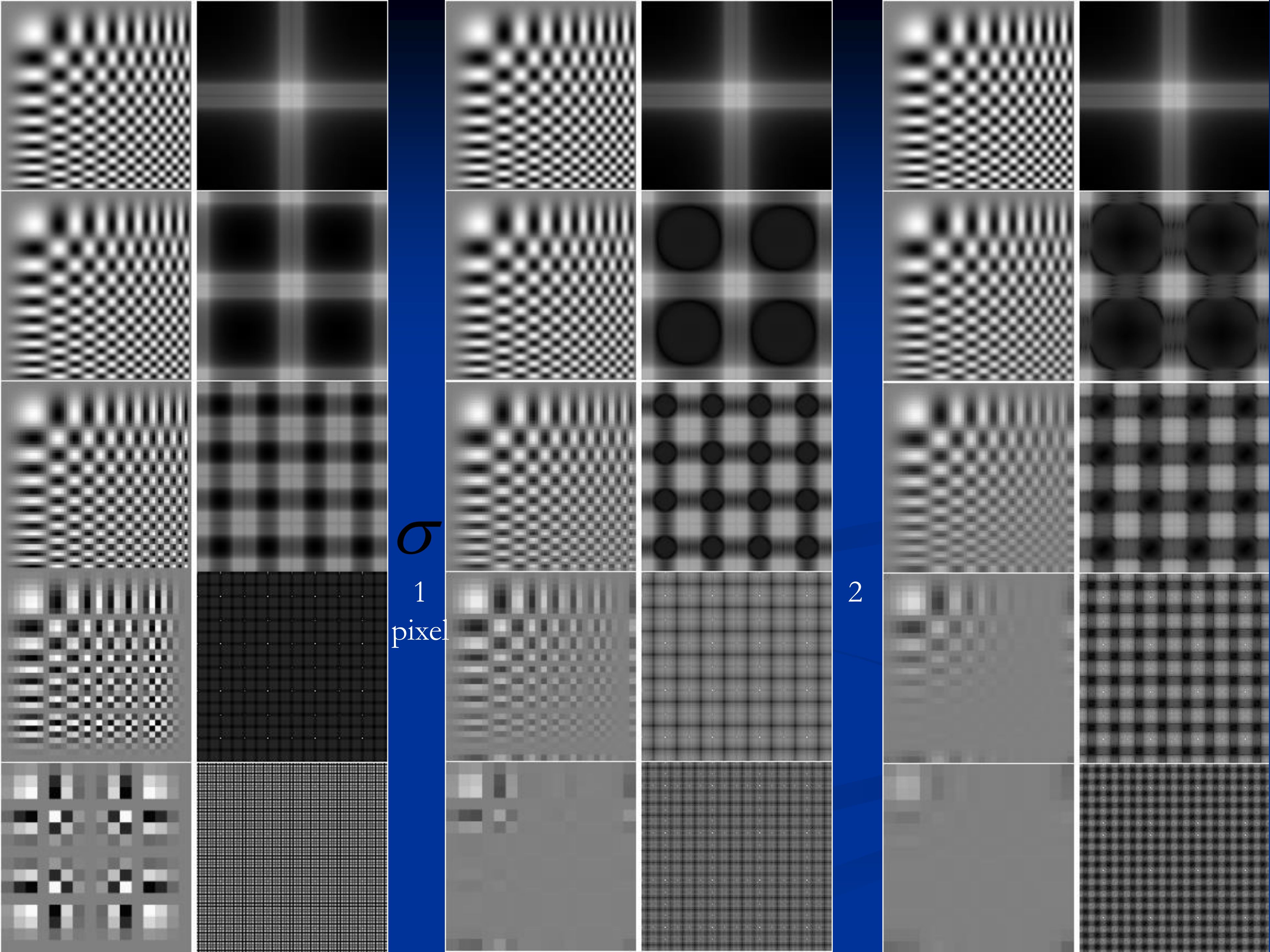
$$F(\omega) \rightarrow F(\omega) * III_{\omega_0}(\omega)$$

Sampling and reconstruction



Sampling and reconstruction

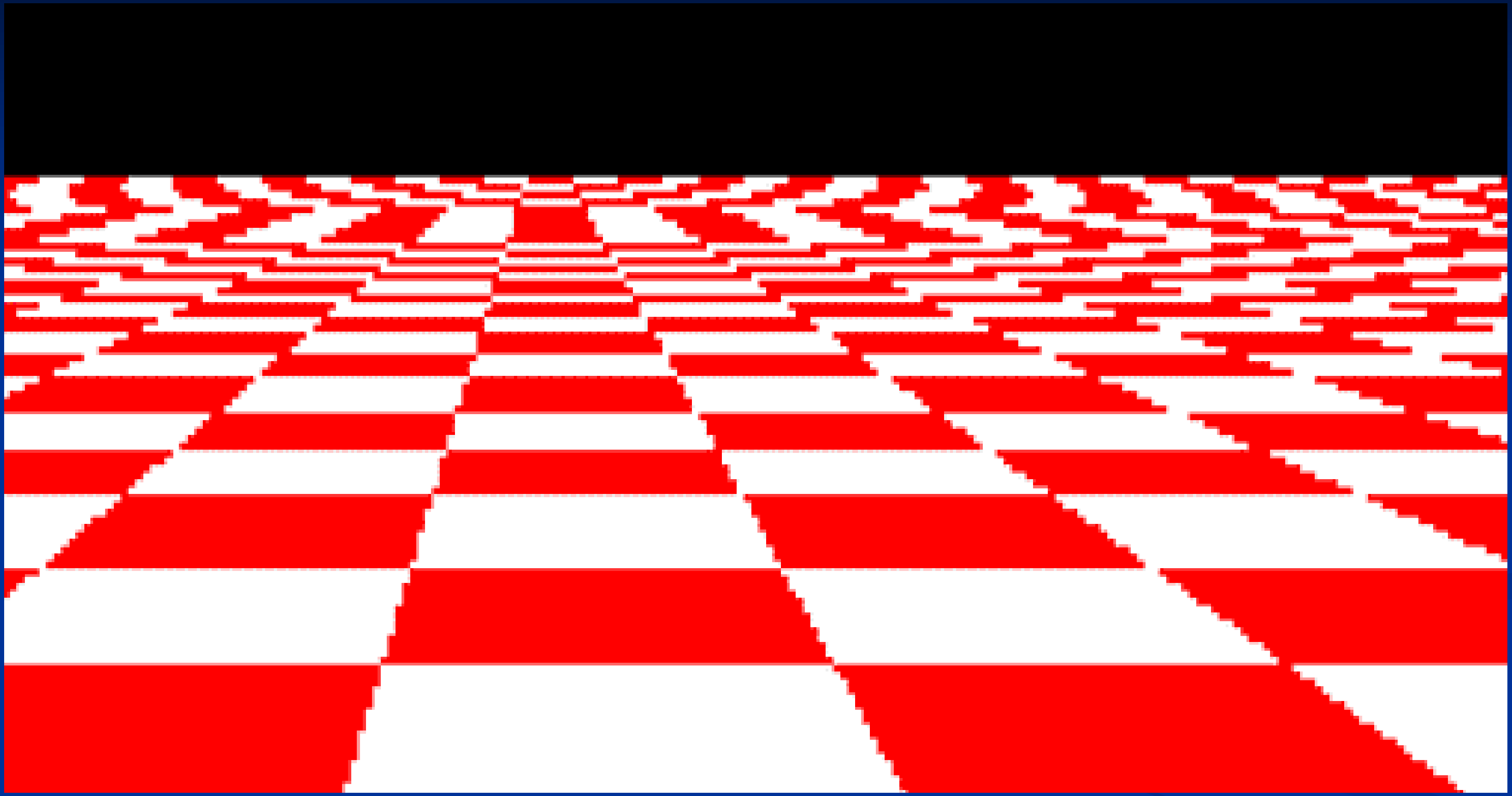



σ
1
pixel
2

Some Aliasing Artifacts

- *Spatial*: Jaggies, Moire
- *Temporal*: Strobe lights, "Wrong" wheel rotations
- *Spatio-Temporal*: Small objects appearing and disappearing

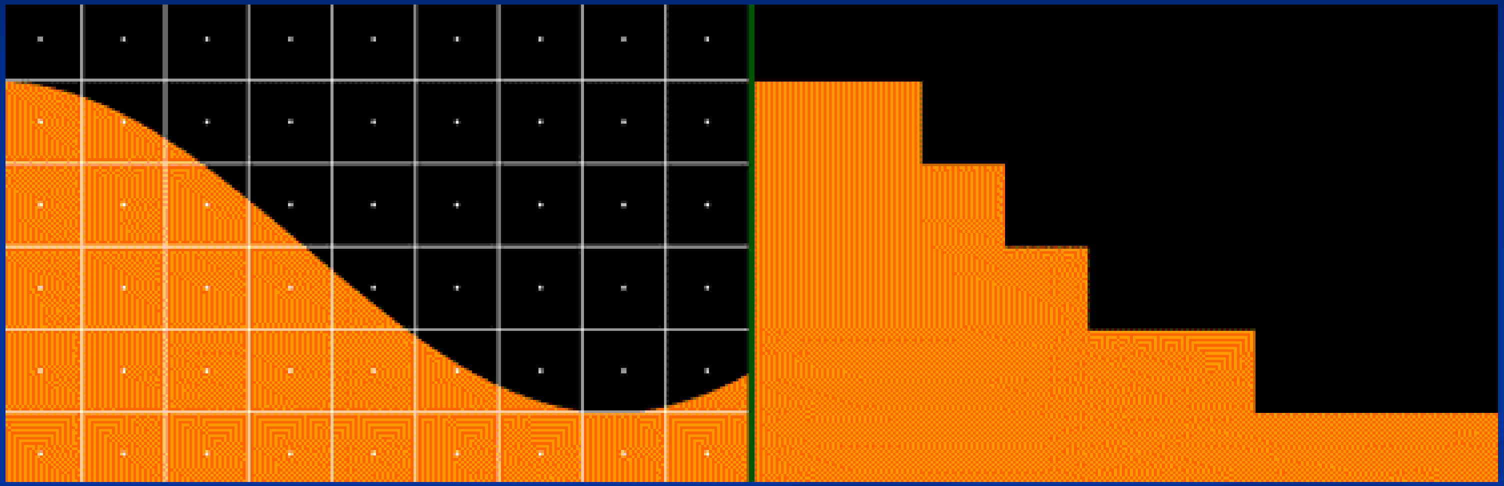
Disintegrating Texture



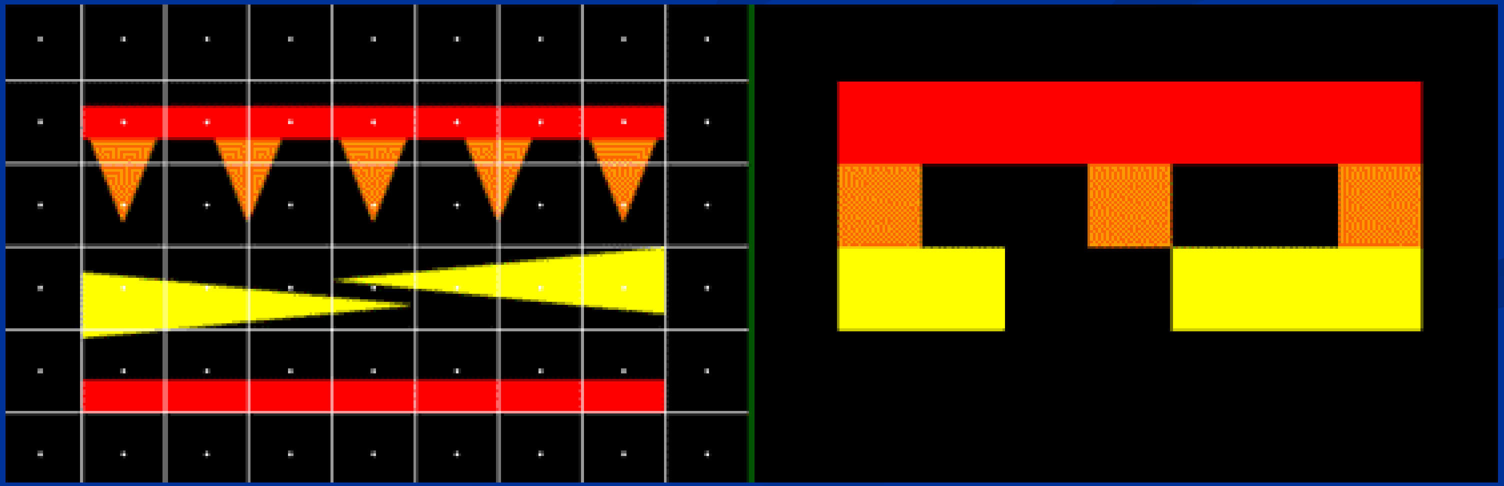
- The checkers on a plane should become smaller with distance.
- But aliasing causes them to become larger and/or irregular.
- Increasing resolution only moves the artefact closer to the horizont.

Spatial Aliasing

Jagged Profiles

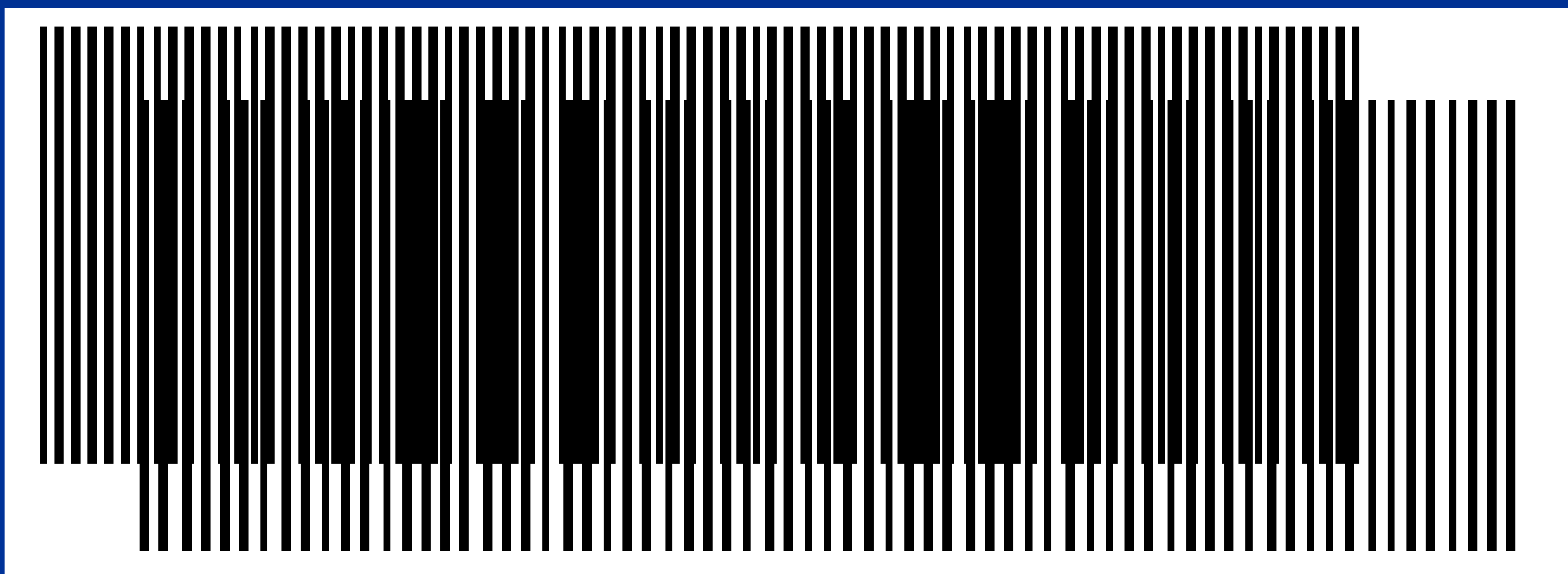


Loss of Detail



Jaggies

Moire Patterns



Moire Patterns